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Chirality and regularization in non-polynomial strong interactions

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Abstract. We discuss the problem of regularization of the non-linear σ model in a fashion consistent with the formal chiral invariance possessed by the non-polynomial lagrangian. The single loop contributions to the Ward–Takahashi identities are only chirally invariant under special conditions on the extra constants arising from a general regularization process we define; these conditions can be satisfied in both the massive and massless cases. We give a natural method of regularization which, however, violates chirality, and indicate that, in particular, superpropagator methods for discussing this or similar theories must be treated with great care to avoid violation of Ward–Takahashi identities.

1. Introduction

Non-polynomial strong interactions have been proposed (for example, Weinberg 1967) as a way of incorporating current algebra structure into a field theoretic mould. Calculations have been made from suitable non-polynomial lagrangians only in the tree approximation, though there have been some tentative attempts (Keck and Taylor 1971, Lehmann and Trute 1972, Bessis and Zinn-Justin 1971, Lazarides and Patani 1971) to include unitary corrections to this. Since these models are non-renormalizable in perturbation theory, they contain extra parameters beyond those in the original lagrangian. The number of these parameters depends heavily on the method of regularization used to obtain finite results in perturbation theory. However the process of regularization need not be consistent with physical properties naturally required of any physical theory. The problem of ensuring unitarity and causality has been recently analysed (Taylor 1972, Daniell and Mitter 1971, Lehmann and Pohlmeyer 1971) in various ways for a class of non-renormalizable theories. This analysis does not cover the derivative coupling arising in the strong interactions we wish to discuss, though it very likely could be extended to this case.

We have, however, a new problem arising since we now wish to regularize in a manner also consistent with whatever chiral invariance is assumed for the lagrangian. Thus if we start with a lagrangian invariant under chiral $SU_2 \times SU_2$ we would hope to preserve the conservation of the vector current V^μ and axial current A^μ. This need not occur as is known in the analogous case of the triangle anomaly (Adler 1969) in quantum electrodynamics, which arises from subtleties in the divergent triangle graph. It is necessary to investigate whether similar anomalies arise in the strong interaction case. That is the main purpose of this paper, at the same time investigating the amount of ambiguity that is present. In particular we will be concerned with any possible reduction in the amount of ambiguity arising from the conservation conditions.

Our investigation of these questions will be restricted here to lowest order closed loop diagrams. We will find already at this level that anomalies may be present. These are restricted however to the massless (chiral symmetric) case, the Ward identity corresponding to PCAC in the massive case being enforceable.

We proceed in § 2 with a discussion of the perturbation theory calculations for the axial vector current in the non-linear σ model. In § 3 we describe our regularization, attempting to use the most general approach possible. In § 4 we analyse the PCAC Ward identity, showing how it constrains the relevant Green functions, and in § 5 the vector current Ward identity. In § 6 we discuss unitarity. In § 7 we consider the massless case. The previous discussion covers this, but here we use a different regularization, inapplicable in the massive case, and show how this leads to incompatible constraints.

2. Perturbation theory rules

The σ model lagrangian is (Gell-Mann and Levy 1960)

$$\mathscr{L} = \tfrac{1}{2}\{(\partial\phi)^2+(\partial\sigma)^2\}+m^2\sigma/\lambda$$

$$= \tfrac{1}{2}\{(\partial\phi)^2-m^2\phi^2\}+\tfrac{1}{2}\lambda^2\sum_{n\geqslant 0}(\lambda^2\phi^2)^n(\phi\partial\phi)^2-\tfrac{1}{2}m^2\lambda^2\sum_{n\geqslant 2}\frac{\Gamma(n-\frac{1}{2})}{n!\Gamma(\frac{1}{2})}(\lambda^2\phi^2)^n \tag{1}$$

where $\sigma^2 = \lambda^{-2}-\phi^2$. We use ϕ as field coordinates, though this is unnecessary. The axial current is

$$A^\mu = \phi^\mu\sigma-\sigma^\mu\phi = -\frac{1}{2\lambda}\phi^\mu\sum_{n\geqslant 0}\frac{\Gamma(n+\frac{5}{2})}{n!\Gamma(\frac{1}{2})}(\lambda^2\phi^2)^n+\phi\phi^\mu\sum_{n\geqslant 0}\frac{\Gamma(n+\frac{3}{2})}{n!\Gamma(\frac{1}{2})}(\lambda^2\phi^2)^n \tag{2}$$

corresponding to transformations generated by

$$\phi' = \sigma = -\frac{1}{2\lambda}\sum_{n\geqslant 0}\frac{\Gamma(n+\frac{5}{2})}{n!\Gamma(\frac{1}{2})}(\lambda^2\phi^2)^n, \tag{3}$$

and with divergence

$$\partial A = -\lambda m^2\phi. \tag{4}$$

That ϕ is an (isospin) triplet of fields is essential—if it were a singlet, $\mathscr{L}(m^2 = 0)$ would be equivalent (through a coordinate transformation) to a free lagrangian. In that case the divergences arising from closed loops will have to cancel. They do not do so when isospin is present, as will be evident from our further discussion.

The Feynman–Dyson expansion of each Green function (GF) contains divergences of arbitrarily high powers of momentum cut-off. Therefore counterterms will involve fields differentiated an arbitrarily large number of times, and be of arbitrarily high degree in the fields. Correspondingly, each GF is not fixed until an infinite number of its values are given. We wonder whether imposing the Ward identities (WI) derived from the PCAC condition and vector current conservation on the GF reduces this to a finite number.

The addition to the lagrangian of counterterms depending on derivatives of the fields modifies the field's conjugate momentum. With the usual renormalizable lagrangians this is simply multiplication by a constant, and easily handled. But in the present case, we do not even know how to find canonical variables once we have added some typical counterterms. If one ignores this problem, the GF one obtains obey formally a complicated equation which does not appear to arise from field equations and commutation relations. One consequence is that in the symmetric ($m = 0$) case, to add only symmetric counterterms is no guarantee of symmetric results.

Consider the calculation of GF involving one current as well as canonical fields. Apart from tadpoles, the simplest (least order in λ and least number of legs) of diagrams with closed loops are those of order 3 with 3 legs, as shown in figure 1, (omitting tadpoles, which do not affect the point to be made). The label f for finite in the loop of the first indicates that the counterterm to the single loop contribution to the elastic two-particle scattering amplitude of figure 2 has been added to the lagrangian. This counterterm involves derivatives of the fields. Therefore $A^\mu = (\partial\mathscr{L}/\partial\phi_\mu)\phi'$ is altered, so giving the third diagram in figure 1. In more tractable theories this would cancel the divergence of the second diagram. That is, in obtaining finite GF of the canonical fields one would also obtain finite GF involving in addition one current. This does not happen in the present case, which may be related to the discussion in the last paragraph but one.

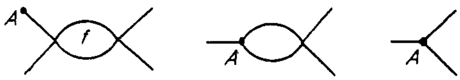

Figure 1. The simplest single loop diagrams for the axial current matrix element.

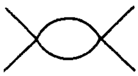

Figure 2. The origin of a lagrangian counterterm.

The PCAC WI are ($+0$ denote time ordered vacuum expectation values):

$$\partial_\mu\langle A_x^{a\mu}\phi_{x_1}^{a_1}\ldots\phi_{x_n}^{a_n}\rangle_{+0} = -\lambda m^2\langle\phi_x^a\phi_{x_1}^{a_1}\ldots\phi_{x_n}^{a_n}\rangle_{+0} - \mathrm{i}\sum_{r=1}^{n}\delta_{xx_r}\langle\phi_x'^a\phi_{x_1}^{a_1}\ldots\phi_{x_n}^{a_n}\rangle_{+0}. \qquad (5)$$

To impose them, one needs to define GF containing ϕ'_x once. Counterterms in ϕ' will contain derivatives of the fields, certainly changing it from the simple initial $\phi' = \sigma$, and presumably changing the group of the transformations.

Thus there are three sets of counterterms involved: those to be added to $\mathscr{L}$, A^μ and ϕ'. The relation $A^\mu = (\partial\mathscr{L}/\partial\phi_\mu)\phi'$ might be used to give one set in terms of the other two. Some restriction on the ϕ' counterterms might be derived from asking something of the corresponding group, though this is not a plausible or attractive proposition.

3. General regularization

Our procedure involves abandoning the use of explicit counterterms in the lagrangian. We have calculated various diagrams formally as products of Δ† and its derivatives Δ_μ, $\Delta_{\mu\nu}$, ... Such an expression has an unambiguous meaning as a distribution on test functions which vanish sufficiently fast when various arguments coincide, and may be extended to test functions without these restrictions. We try to fix the extension in each

† $\Delta = -\mathrm{i}(\Box + m^2)^{-1}$, $\Box = \partial_\mu\partial^\mu$.

case with the WI. In place of the mentioned connection between the three sets of counterterms, we assume that wherever the same product of $\Delta, \Delta_\mu, \ldots$ occurs, for example in different terms of a Ward identity, it takes the same value (ie has the same extension).

To check the WI corresponding to PCAC formally, that is with divergences untreated (or no counterterms), one needs to take proper account of tadpoles. Particularly, if one uses $-\Delta_\mu(x-y)$ for $\langle\phi_\mu(x)\phi_2(y)\rangle_{+0}$, one must add a term of the form $\mathrm{i}\delta^4(0)\ln F(\phi)$, which contains tadpoles, to the lagrangian. Having detached our procedure from its formal background we no longer have a rational way of dealing with tadpoles; we just delete them (and $\mathrm{i}\delta^4(0)\ln F(\phi)$) completely. The omitted terms—tadpoles and $-\mathrm{i}\delta(x-y)$—have similar form, and might be expected to cancel each other in any case, though we will not attempt to show that here. This is accompanied by the rule that in a product containing $\Delta^{\mu\mu}(x-y)$ and at least one other line between x and y we replace $\Delta^{\mu\mu}$ by $-m^2\Delta$. We find that this enables us to verify formally the WI in our single loop approximation.

Having formally verified a WI in some order, we then examine what deviation one finds upon looking more closely. Thus we may have used the formal identity

$$(\Delta^2)_\mu = 2\Delta\Delta_\mu. \tag{6}$$

In fact having defined Δ^2 and $\Delta\Delta_{,\mu}$ we find

$$(\Delta^2)_\mu = 2\Delta\Delta_\mu + a\delta_\mu \tag{7}$$

where a is a constant depending on the extension of Δ^2 and $\Delta\Delta_\mu$. The WI impose constraints on a and several analogues. We find that these can be satisfied by choice of the extensions, involving two free parameters. The special regularization in the massless case is similar, and is described in § 7.

4. The regularized axial current Ward identity

We carry out our calculation for the third order with three legs (ie $n = 3$). The WI is shown diagrammatically in figure 3. We omit λ hereafter.

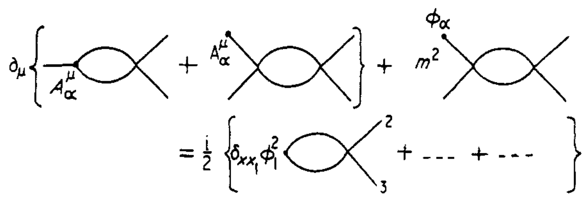

Figure 3. The formal Ward identity.

The formal verification may be greatly simplified by notation, which we now take a page to explain. We write figure 4 as

$$\frac{\mathrm{i}}{2}\frac{\delta}{\delta\phi_1}\frac{\delta}{\delta\phi_2}\frac{\delta}{\delta\phi_3}\frac{\delta A^\alpha_\mu}{\delta\phi_\beta\,\delta\phi_\gamma}\frac{\delta\mathcal{L}}{\delta\phi_\beta\,\delta\phi_\gamma}$$

$$= \frac{\mathrm{i}}{2}\frac{\delta A^\alpha_\mu}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_1}\frac{\delta\mathcal{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3} + \text{two permutations of } 1, 2, 3. \tag{8}$$

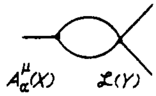

Figure 4. The contribution to an axial current matrix element.

Here $\phi_1 = \phi(x_1)$, etc. β and γ correspond to the internal lines — to two factors of $\Delta(x-y)$ that are understood. Factors of $\Delta(x-x_1)$, $\Delta(y-x_2)$ and $\Delta(y-x_3)$, and integration over y are understood. From

$$A^{\mu} = \phi^{\mu} - \tfrac{1}{2}\phi^{\mu}\phi^2 + \phi(\phi\phi^{\mu}) + \text{terms of higher degree} \tag{9}$$

we have

$$\frac{\delta A^{\alpha\mu}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_1} = -(\delta^{\alpha}_{1\mu}\,\delta^{\beta}_{\gamma} + \delta^{\alpha}_{\beta\mu}\,\delta^{\gamma}_{1} + \delta^{\alpha}_{\gamma\mu}\,\delta^{\beta}_{1}) + \delta^{\alpha}_{1}(\delta^{\beta\mu}_{\gamma} + \delta^{\beta}_{\gamma\mu}) + \delta^{\alpha}_{\beta}(\delta^{\gamma\mu}_{1} + \delta^{\gamma}_{1\mu}) + \delta^{\alpha}_{\gamma}(\delta^{\beta\mu}_{1} + \delta^{\beta}_{1\mu}). \tag{10}$$

From

$$\mathcal{L} = \tfrac{1}{2}(\phi\partial\phi)^2 - \tfrac{1}{8}m^2\phi^4 \tag{11}$$

we have

$$\frac{\delta\mathcal{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3} = \delta^{\beta\nu}_{\gamma}\,\delta^{2\nu}_{3} + \text{eleven permutations of } \beta, \gamma, 2, 3$$
$$-m^2\,\delta^{\beta}_{\gamma}\,\delta^{2}_{3} + \text{two permutations} \tag{12}$$

(whether a letter is a sub- or super-script is not significant).

To explain the meaning of these formulae, take the first terms from (10) and (12): in (8) they give

$$-\frac{\mathrm{i}}{2}\cdot\delta^{\alpha}_{1\mu}\,\delta^{\beta}_{\gamma}\,.\,\delta^{\beta\nu}_{\gamma}\,\delta^{2\nu}_{3}. \tag{13}$$

Here μ stands for $\partial/\partial x^{\mu}$, ν for $\partial/\partial y^{\nu}$, α for the component of A^{μ}, β, γ for internal lines and 1, 2, 3 for external lines. μ written next to 1 means that the implicit factor $\Delta(x-x_1)$ occurs differentiated with respect to x. ν next to β means that the implicit factor $\Delta(x-y)$ corresponding to β occurs differentiated with respect to y. ν next to 2 means that the implicit factor $\Delta(y-x_2)$ occurs differentiated with respect to y. Thus (13) stands for

$$-\frac{\mathrm{i}}{2}\int \mathrm{d}y\,\Delta^{\mu}(x-x_1)\Delta^{\nu}(y-x)\Delta(x-y)\Delta^{\nu}(y-x_2)\Delta(y-x_3)\,\delta^{\alpha}_{1}\,\delta^{\beta}_{\gamma}\,\delta^{\beta}_{\gamma}\,\delta^{2}_{3} \tag{14}$$

where now δ^{α}_{1} etc are simply Kronecker deltas, and repeated indices are summed over.

Now we proceed with the verification of the WI. Figure 5 is equal to

$$\frac{\mathrm{i}}{2}\,\frac{\delta\mathcal{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_{\alpha'}\,\delta\phi_1}\,\frac{\delta\mathcal{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3} + \text{two permutations of 1, 2, 3.} \tag{15}$$

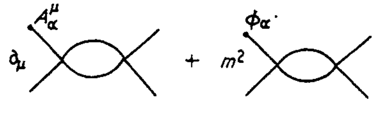

Figure 5. Terms of a Ward identity, to order λ^3.

Here the prime in $\phi_{\alpha'}$ means that by partial integration all derivatives are transferred from α to β, γ and 1; that is,

$$\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_\alpha\,\delta\phi_1} = (\delta_1^{\alpha\mu}+\delta^\alpha_{1\mu})(\delta_\gamma^{\beta\mu}+\delta^\beta_{\gamma\mu})+(\delta_\beta^{\alpha\mu}+\delta^\alpha_{\beta\mu})(\delta_1^{\gamma\mu}+\delta^\gamma_{1\mu})$$
$$+(\delta_\gamma^{\alpha\mu}+\delta^\alpha_{\gamma\mu})(\delta_1^{\beta\mu}+\delta^\beta_{1\mu})-m^2(\delta^\alpha_1\,\delta^\beta_\gamma+\delta^\alpha_\beta\,\delta^\gamma_1+\delta^\alpha_\gamma\,\delta^\beta_1) \tag{16}$$

$$\to \frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_{\alpha'}\,\delta\phi_1} = -\delta^\alpha_1(\delta_\gamma^{\beta\mu\mu}+2\delta^{\beta\mu}_{\gamma\mu}+\delta^\beta_{\gamma\mu\mu})-\delta^\alpha_\beta(\delta_1^{\gamma\mu\mu}+2\delta^{\gamma\mu}_{1\mu}+\delta^\gamma_{1\mu\mu})$$
$$-\delta^\alpha_\gamma(\delta_1^{\beta\mu\mu}+2\delta^{\beta\mu}_{1\mu}+\delta^\beta_{1\mu\mu})-m^2(\delta^\alpha_1\,\delta^\beta_\gamma+\delta^\alpha_\beta\,\delta^\gamma_1+\delta^\alpha_\gamma\,\delta^\beta_1). \tag{17}$$

We make another convention: $\delta^\alpha_\beta\,\delta^\gamma_1|_{\delta_1}$ means that the factor $\Delta(x-x_1)$ is replaced by $\delta(x-x_1)$. As mentioned in § 3, we put $\delta_1^{\beta\mu\mu} = -m^2\,\delta^\beta_1$ etc (but $\delta^\beta_{1\mu\mu} = -m^2\,\delta^\beta_1-\mathrm{i}\delta^\beta_1|_{\delta_1}$). Thus

$$\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_{\alpha'}\,\delta\phi_1} = -2(\delta^\alpha_1\,\delta^{\beta\mu}_{\gamma\mu}+\delta^\alpha_\beta\,\delta^{\gamma\mu}_{1\mu}+\delta^\alpha_\gamma\,\delta^{\beta\mu}_{1\mu})+m^2(\delta^\alpha_1\,\delta^\beta_\gamma+\delta^\alpha_\beta\,\delta^\gamma_1+\delta^\alpha_\gamma\,\delta^\beta_1)+\mathrm{i}(\delta^\alpha_\beta\,\delta^\gamma_1+\delta^\alpha_\gamma\,\delta^\beta_1)|_{\delta_1}. \tag{18}$$

Returning to the first term in the WI (figure 3), notice that in placing the divergence, $\partial/\partial x^\mu$ acts only on 1, β and γ and

$$\partial_\mu\frac{\partial A^{\alpha\mu}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_1} = 2(\delta^\alpha_1\,\delta^{\beta\mu}_{\gamma\mu}+\delta^\alpha_\beta\,\delta^{\gamma\mu}_{1\mu}+\delta^\alpha_\gamma\,\delta^{\beta\mu}_{1\mu})-m^2(\delta^\alpha_1\,\delta^\beta_\gamma+\delta^\alpha_\beta\,\delta^\gamma_1+\delta^\alpha_\gamma\,\delta^\beta_1)$$
$$+\mathrm{i}(\delta^\alpha_1\,\delta^\beta_\gamma-\delta^\alpha_\beta\,\delta^\gamma_1-\delta^\alpha_\gamma\,\delta^\beta_1)|_{\delta_1}. \tag{19}$$

Therefore the left-hand side of figure 3 is equal to

$$-\tfrac{1}{2}\delta^\alpha_1\,\delta^\beta_\gamma|_{\delta_1}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two permutations of 1, 2, 3} \tag{20}$$

and the WI follows immediately.

We show in the appendix that the same approach works for the single loop term of the WI involving $\langle A^\mu\phi_1\phi_2\phi_3\phi_4\phi_5\rangle_{+0}$.

Now we ask what deviations from the WI (figure 3) are caused by the defects in the various formal identities used. We define figures 6(*a*), (*b*), (*c*) and (*d*) as respectively

$$\frac{\mathrm{i}}{2}\frac{\partial A^{\mu\alpha}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_1}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two permutations} \tag{21a}$$

$$-\frac{1}{2}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_{\alpha\mu}\,\delta\phi_1}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two} \tag{21b}$$

$$-\frac{1}{2}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_\alpha\,\delta\phi_1}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two} \tag{21c}$$

$$-\frac{\mathrm{i}}{2}\frac{\delta\phi_1^2}{\delta\phi_\beta\,\delta\phi_\gamma}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3} \tag{21d}$$

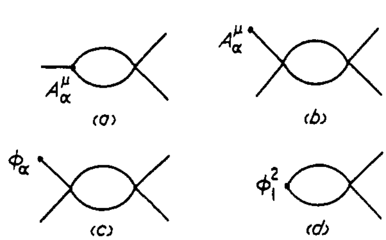

Figure 6. Sources of corrections to the formal Ward identity.

now with the products of $\Delta, \Delta_\mu, \ldots$ finite, though undetermined within addition of a polynomial to the Fourier transform. Our deviations arise in two places. Firstly, in the divergence of the first term, of

$$\tfrac{1}{2}\mathrm{i}\{-(\delta^\alpha_{1\mu}\,\delta^\beta_\gamma+\delta^\alpha_{\beta\mu}\,\delta^\gamma_1+\delta^\alpha_{\gamma\mu}\,\delta^\beta_1)+\delta^\alpha_1(\delta^{\beta\mu}_\gamma+\delta^\beta_{\gamma\mu})+\delta^\alpha_\beta(\delta^{\gamma\mu}_1+\delta^\gamma_{1\mu})+\delta^\alpha_\gamma(\delta^{\beta\mu}_1+\delta^\mu_{1\mu})\}$$
$$\times\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two permutations.} \tag{22}$$

Secondly they arise from the second and third terms, not in the divergence, which simply amputates the α leg, but in the subsequent integration by parts, where we take $\partial/\partial x^\mu$ on

$$-\frac{\mathrm{i}}{2}\{\delta^\alpha_1(\delta^{\beta\mu}_\gamma+\delta^\beta_{\gamma\mu})+\delta^\alpha_\beta(\delta^{\gamma\mu}_1+\delta^\gamma_{1\mu})+\delta^\alpha_\gamma(\delta^{\beta\mu}_1+\delta^\beta_{1\mu})\}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two.} \tag{23}$$

So the difference between the left-hand side and the right-hand side of the WI (figure 5) is the deviation from formality of $\partial/\partial x^\mu$ on

$$\begin{aligned}
&-\frac{\mathrm{i}}{2}\{\delta^\alpha_{1\mu}\,\delta^\beta_\gamma+\delta^\alpha_{\beta\mu}\,\delta^\gamma_1+\delta^\alpha_{\gamma\mu}\,\delta^\beta_1\}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3}+\text{two permutations}\\
&\quad= -\mathrm{i}[\Delta^2\,\delta^\alpha_{1\mu}(\delta^{2\nu}_{3\nu}-\tfrac{5}{2}m^2\,\delta^2_3)-\Delta\Delta_\nu\,\delta^\alpha_{1\mu}(\delta^{2\nu}_3+\delta^2_{3\nu})\\
&\qquad+\Delta\Delta_\mu\{\delta^\alpha_{2\nu}\,\delta'_{3\nu}+\delta^\alpha_{3\nu}\,\delta'_{2\nu}-\tfrac{1}{2}m^2(\delta^\alpha_1\,\delta^2_3+\delta^\alpha_2\,\delta'_3+\delta^\alpha_3\,\delta'_2)\}\\
&\qquad+\Delta_\nu\,\Delta_\nu\,\delta^\alpha_{1\mu}\,\delta^2_3-\Delta_\mu\,\Delta_\nu\{\delta^\alpha_1(\delta^{2\nu}_3+\delta^2_{3\nu})+\delta^\alpha_{2\nu}\,\delta'_3+\delta^\alpha_{3\nu}\,\delta'_2\}\\
&\qquad-\Delta\,\Delta_{\mu\nu}\{\delta^\alpha_1(\delta^{2\nu}_3+\delta^2_{3\nu})+\delta^\alpha_2\,\delta'_{3\nu}+\delta^\alpha_3\,\delta'_{2\nu}\}+\Delta_\nu\,\Delta_{\mu\nu}(\delta^\alpha_2\,\delta'_3+\delta^\alpha_3\,\delta'_2)]+\text{two.}
\end{aligned} \tag{24}$$

Here we have written the internal lines, corresponding to β and γ, explicitly, and adjusted the signs so that Δ_ν means $(\partial/\partial x^\nu)\,\Delta(x-y)$, although ν next to 2 or 3 still means $\partial/\partial y^\nu$.

The errors in the formal identities must vanish on the restricted test functions used initially in defining the distributions involved, so we have

$$\begin{aligned}
&\delta(\Delta^2)_\mu \equiv (\Delta^2)_\mu - 2\Delta\Delta_\mu = a\,\delta_\mu\\
&\delta(\Delta\Delta_\nu)_\mu = \{(b\Box+c)g_{\mu\nu}+d\partial_{\mu\nu}\}\,\delta\\
&\delta(\Delta_\nu\Delta_\nu)_\mu = (e\Box+f)\,\delta_\mu\\
&\delta(\Delta_\mu\Delta_\nu)_\mu \equiv (\Delta_\mu\Delta_\nu)_\mu - \Delta_\mu\Delta_{\mu\nu} = (g\Box+h)\,\delta_\nu\\
&\delta(\Delta\Delta_{\mu\nu})_\mu = (j\Box+k)\,\delta_\nu\\
&\delta(\Delta_\nu\Delta_{\mu\nu})_\mu = (l\Box^2+m\Box+n)\,\delta.
\end{aligned} \tag{25}$$

Then the coefficient of $\delta_1^\alpha\, \delta_3^2$ in the Fourier transform,

$$\prod_{k=1}^{3} \int \mathrm{d}X_n \exp(\mathrm{i}P_n X_n),$$

at $X = 0$, with legs 1, 2 and 3 amputated, of the legs of the WI is

$$\begin{aligned}
-\mathrm{i}\Bigg\{ & -P_{23}^4(g+j)+(P_{12}^4+P_{13}^4)\left(-\frac{4b+d}{2}-\frac{g+j}{2}+l\right)+(P_{12}^2+P_{13}^2)P_{23}^2\left(\frac{a}{4}+\frac{e-b-d}{2}\right)\\
& +P_{23}^2P_1^2\left(-\frac{a}{2}+b+d-e\right)+P_{23}^2(P_2^2+P_3^2)\left(-\frac{a}{4}+\frac{b+d-e}{2}\right)\\
& +(P_{12}^2+P_{13}^2)P_1^2\left(\frac{4b+d}{2}+\frac{j-g}{2}\right)+(P_{12}^2P_3^2+P_{13}^2P_2)\left(-\frac{a}{4}\right)\\
& +(P_{12}^2P_2^2+P_{13}^2P_3^2)\left(-\frac{a}{4}+\frac{4b+d}{2}+\frac{g-j}{2}\right)+(P_2^4+P_3^4)\frac{a}{4}\\
& +P_{23}^2\left(-\frac{4b+d}{2}m^2+h+k\right)\\
& +(P_{12}^2+P_{13}^2)\left(\tfrac{5}{4}m^2a-m^2\frac{4b+d}{2}+\tfrac{1}{2}f+\tfrac{5}{2}c+\tfrac{1}{2}h+\tfrac{1}{2}k-m\right)\\
& +P_1^2(-\tfrac{5}{2}am^2+f-5c+h-k)\\
& +(P_2^2+P_3^2)(-\tfrac{5}{4}m^2a+\tfrac{1}{2}f-\tfrac{5}{2}c-\tfrac{1}{2}h-\tfrac{1}{2}k)+n\Bigg\},
\end{aligned} \tag{26}$$

where $P_{23} = P_2+P_3$ etc, P_1^2, P_2^2, P_3^2, P_{23}^2, P_{12}^2 and P_{13}^2 are independent. This vanishes if and only if all but b, c, d, and e vanish, and $d = -4b$, $e = -12b$. Is this possible? We can answer this without looking in detail at our distributions, denoting one possible choice of Δ^2 by $(\Delta^2)_0$, etc, corresponding to $a_0, b_0, \ldots,$ by looking at the effect of the change to

$$\begin{aligned}
\Delta^2 &= (\Delta^2)_0 + A\,\delta\\
\Delta\Delta_\mu &= (\Delta\Delta_\mu)_0 + B\,\delta_\mu\\
\Delta_\mu\Delta_\nu &= (\Delta_\mu\Delta_\nu)_0 + |(C\Box + D)g_{\mu\nu} + E\partial_{\mu\nu}|\,\delta\\
\Delta\Delta_{\mu\nu} &= (\Delta\Delta_{\mu\nu})_0 + |(F\Box + G)g_{\mu\nu} + H\partial_{\mu\nu}|\,\delta\\
\Delta_\nu\Delta_{\mu\nu} &= (\Delta_\nu\Delta_{\mu\nu})_0 + (J\Box + K)\,\delta_\mu\\
\Delta_{\mu\nu}\Delta^{\mu\nu} &= (\Delta_{\mu\nu}\Delta^{\mu\nu})_0 + (L\Box^2 + M\Box + N)\,\delta.
\end{aligned} \tag{27}$$

From

$$\delta(\Delta^2)_\mu = \delta(\Delta^2)_{\mu 0} + (A-2B)\,\delta_\mu \tag{28}$$

etc, we find

$$\begin{aligned}
a &= a_0 + A - 2B\\
b &= b_0 - C - F\\
c &= c_0 - D - G\\
d &= d_0 + B - E - H
\end{aligned}$$

$$
\begin{aligned}
e &= e_0+4C+E-2J \\
f &= f_0+4D-2K \\
g &= g_0+C+E-J \\
h &= h_0+D-K+m^2B \\
j &= j_0+F+H-J \\
k &= k_0+G-K+m^2B \\
l &= l_0+J-L \\
m &= m_0+K-M+m^2(4C+E) \\
n &= n_0-N+4m^2D.
\end{aligned} \tag{29}
$$

If $m \neq 0$, these equations have a unique solution for $A, B, \ldots$ in terms of $a, a_0, b, b_0, \ldots$ (whatever they are), so we can enforce the WI. The parameters b and c are unconstrained. If $m = 0$, the equations can be solved if and only if $c+f-3h+k = c_0+f_0-3h_0+k_0$. The WI requires that $c+f-3h+k = c$, so fixes c. But now the solution is not unique, and our theory still has two free parameters, for example, b and B.

A faint hope one might entertain at this point is that the GF are independent of these parameters. They are not, nor is the four particle mass shell S matrix.

Because some of the arguments of the products of $\Delta, \Delta_\mu, \ldots$ are integrated over, one can formally integrate by parts and change variables of integration to obtain different expressions for the same GF. This will presumably amount to adding any local expression with appropriate properties. It is not difficult to see that allowing this will largely undo the above constraints on the distributions.

One can carry out a similar calculation for the WI involving $\langle A^\mu \phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \rangle_{+0}$. Being extremely complicated and giving many opportunities for arithmetical errors, we have not completed it. We see no reasons in principle why one could not apply our method to all the WI.

5. Regularized vector current Ward identity

This does not affect these conclusions. The distributions considered above occur in the typical diagram of figure 7.

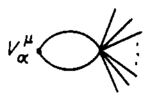

Figure 7. A diagram occurring in a vector current Ward identity.

The WI requires

$$\partial_\mu(\Delta_\mu \Delta_\nu - \Delta\Delta_{\mu\nu}) = 0 \tag{30}$$

so

$$g-j = h-k = 0. \tag{31}$$

This is already required by the axial current WI. Thus there are no further restrictions on the two arbitrary constants.

6. Unitarity

The formal identities used for the products of $\Delta, \Delta_\mu, \ldots$ are all true if $\Delta, \Delta_\mu, \ldots$ are replaced by $\Delta^-, \Delta_\mu^-, \ldots$, so for unitarity it is sufficient that

$$\begin{aligned} \mathrm{Re}\,\Delta^2 &= \Delta^{-2} \\ \mathrm{Re}\,\Delta\Delta_\mu &= \Delta^-\Delta_\mu^- \end{aligned} \tag{32}$$

etc. We need now to give some details of our distributions. Let

$$\begin{aligned} \Delta^{-2} &= \mathscr{A}^- \\ \Delta^-\Delta_\mu^- &= \mathscr{B}^-_{,\mu} \\ \Delta_\mu^-\Delta_\nu^- &= g_{\mu\nu}\mathscr{C}^- + \partial_{\mu\nu}\mathscr{E}^- \end{aligned} \tag{33}$$

etc. Then

$$\begin{aligned} \Delta^2 \equiv \mathscr{A} &= \mathscr{A}^-\theta + (x \to -x) \\ \Delta\Delta_\mu \equiv \mathscr{B}_{,\mu} &= \mathscr{B}^-_{,\mu}\theta + (x \to -x) \end{aligned} \tag{34}$$

etc, modulo infinite polynomials. Now

$$\mathscr{A}^-(x) = \int \mathrm{d}\mu^2 \rho_{\mathscr{A}}(\mu^2)\Delta^-(x, \mu^2) \tag{35}$$

etc, where $\rho_{\mathscr{A}}$ etc are real, so we can fix these polynomials for $(\Delta^2)_0$ etc, corresponding to $\mathscr{A}_0$ etc, by

$$\begin{aligned} \mathscr{A}_0 &= \int \frac{\mathrm{d}\mu^2}{\mu^2} \rho_{\mathscr{A}}(\mu^2)(-\Box)\Delta(x, \mu^2) \\ \mathscr{C}_0 &= \int \frac{\mathrm{d}\mu^2}{\mu^4} \rho_{\mathscr{C}}(\mu^2)(-\Box)^2\Delta(x, \mu^2) \end{aligned} \tag{36}$$

etc. These $(\Delta^2)_0$ etc satisfy the unitarity equations (32), so they hold for Δ^2 etc if and only if $A, B, \ldots$ are imaginary. Also

$$\begin{aligned} \delta_0(\Delta\Delta_\nu)_\mu &= \int \mathrm{d}\mu^2 \left(\frac{\rho_{\mathscr{B}}}{\mu^2}(-\Box\partial_{\mu\nu}) - \frac{\rho_{\mathscr{C}} + \rho_{\mathscr{K}}}{\mu^4}\Box^2 g_{\mu\nu} - \frac{\rho_{\mathscr{E}} + \rho_{\mathscr{K}}}{\mu^2}\partial_{\mu\nu}\Box \right) \Delta(x, \mu^2) \\ &= \int \mathrm{d}\mu^2 \{\rho_{\mathscr{B}}\partial_{\mu\nu} - (\rho_{\mathscr{C}} + \rho_{\mathscr{K}})g_{\mu\nu} + (\rho_{\mathscr{E}} + \rho_{\mathscr{K}})\partial_{\mu\nu}\}\, \Delta(x, \mu^2) \\ &\quad + \mathrm{i}\int \mathrm{d}\mu^2 \left(\frac{\rho_{\mathscr{B}}}{\mu^2}\partial_{\mu\nu} + g_{\mu\nu}\frac{\rho_{\mathscr{C}} + \rho_{\mathscr{K}}}{\mu^4}(\Box - \mu^2) + \frac{\rho_{\mathscr{E}} + \rho_{\mathscr{K}}}{\mu^2}\partial_{\mu\nu} \right) \delta_x. \end{aligned} \tag{37}$$

The first term must vanish because $\delta(\Delta^-\Delta_\nu^-)_\mu = 0$. Thus b_0, c_0 and d_0 are imaginary. It is easy by this method to see that all of $a_0, b_0, \ldots$ are imaginary. If $m = 0$ we should, say, replace $(-\Box/\mu^2)^n$ by $\{(\lambda^2 - \Box)/(\lambda^2 + \mu^2)\}^n$. This gives the same result. It follows that we maintain unitarity if $a, b, \ldots$ are imaginary. From (29) and the following discussion we see that this is done if b and C are made imaginary.

7. Zero mass mesons

The zero mass propagator is

$$\Delta_x = \frac{\kappa}{x^2 - i0}, \qquad \kappa = -\frac{1}{4\pi^2} \tag{38}$$

so formally

$$\begin{aligned} \Delta\Delta_\mu &= -\frac{2}{\kappa} X_\mu \Delta^3 \\ \Delta_\mu \Delta_\nu &= \frac{4}{\kappa^2} X_\mu X_\nu \Delta^4 \\ \Delta\Delta_{\mu\nu} &= -\frac{2}{\kappa} g_{\mu\nu} \Delta^3 + \frac{8}{\kappa^2} X_\mu X_\nu \Delta^4 \\ \Delta_\nu \Delta_{\mu\nu} &= -\frac{12}{\kappa^2} X_\mu \Delta^4 \\ \Delta_{\mu\nu}\Delta_{\mu\nu} &= \frac{48}{\kappa^2}\Delta^4. \end{aligned} \tag{39}$$

Whereas in the previous sections we take the raw products on the left-hand side as unvarying (from term to term) but unrelated (before imposing the WI) components of our formulae, now we take Δ^n. This is stronger, since now there will be relations between the products: consider the effects of the changes

$$\Delta^n = (\Delta^n)_0 + A_n \Box^{n-2} \delta. \tag{40}$$

(The A_n are dimensionless. One could include $A_n' \Box^{n-3} \delta + A_n'' \Box^{n-4} \delta + \ldots$ but leaving them out is consistent and perhaps preferable, the λ of the original lagrangian then fixing alone the scale of the system, and does not affect the argument.) One finds

$$\begin{aligned} A &= A_2 \\ B &= \frac{4}{\kappa} A_3 \\ C &= \tfrac{1}{2}E = \tfrac{1}{2}F + \tfrac{1}{4}B = \tfrac{1}{4}H = \tfrac{1}{3}J = \tfrac{1}{3}L = \frac{16}{\kappa^2} A_4 \\ D &= G = 0. \end{aligned} \tag{41}$$

It follows in particular that g and l, both required to vanish by the WI, are independent of A_n.

Define a_n and b_n by

$$\begin{aligned} \Delta^n_{,\mu} &= -2nX_\mu \Delta^{n+1} + a_n \Box^{n-2} \delta_\mu \\ X^2 \Delta^n &= \kappa \Delta^{n-1} + b_n \Box^{n-3} \delta. \end{aligned} \tag{42}$$

They are not independent:

$$\begin{aligned} a_n &= a_n^0 + A_n - \frac{4}{\kappa} n(n-1) A_{n+1} \\ b_n &= b_n^0 + 4(n-1)(n-2) A_n - \kappa A_{n-1} \end{aligned} \tag{43}$$

so

$$c_n \equiv a_n + \frac{b_{n+1}}{\kappa} \tag{44}$$

is invariant. One finds $g = 128c_4/\kappa^2$, $l = 96c_4/\kappa^2$. c_n can be obtained most easily from

$$(X^\mu \Delta^n)_\mu = -(2n-4)\Delta^n - 2nc_n \Box^{n-2}\delta \tag{45}$$

which gives

$$c_n = -\frac{1}{n}\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho^2} \frac{\Delta^n(\rho)}{(-\rho^2)^{n-2}}. \tag{46}$$

Since

$$\Delta^n(P) = \frac{-\mathrm{i}(\frac{1}{4}\kappa)^{n-1}}{(n-1)!(n-2)!}(-\rho^2)^{n-2}\ln(-\rho^2) \tag{47}$$

except for an added term proportional to $P^{2(n-2)}$ (lower powers omitted for dimensional reasons, as above), we have

$$c_n = -\mathrm{i}\frac{(\frac{1}{4}\kappa)^{n-1}}{n!(n-2)!}. \tag{48}$$

Thus g, $l \neq 0$, so we cannot fulfil the WI.

The possibility of partial integrations and translations, mentioned at the end of § 4, does not arise here. Another merit of this method of regularization is that although anomalies arise here, they do not do so with this method in massless quantum electrodynamics. In that case the polarization operator $\pi^{\mu\nu}$, for the photon, in the single loop approximation has the value

$$\pi^{\mu\nu} = -\frac{24}{\kappa}\Delta^3 \delta_{\mu\nu} + \frac{48}{\kappa^2} X_\mu X_\nu \Delta^4. \tag{49}$$

Then

$$\begin{aligned} \pi^{\mu\nu}_{,\nu} &= 12\{(j-g)\Box + (k-h)\}\delta_\mu \\ &= 12\left(\frac{128}{k^2}c_4 - \frac{2}{k}a_3\right)\Box\,\delta_\mu. \end{aligned} \tag{50}$$

This is zero if and only if

$$a_3 = \frac{64}{k}c_4 \tag{51}$$

so this method is successful.

8. Discussion

In summary, we have shown that the Ward identities can be satisfied in the single closed loop approximation, in both massive and massless theories. This is true provided a general enough regularization scheme is used, allowing the introduction of a suitable number of arbitrary constants. Our results show that the ambiguity in the most general regularization of the single loop contribution is indeed reduced by the imposition of chirality. Without the invariance condition there are thirteen arbitrary constants; after the Ward identities have been achieved there are only two free parameters left. Unitarity reduces these even further to two real parameters.

An alternative regularization scheme, with less arbitrariness, is possible in the massless case, but does not preserve chiral invariance. It may be possible to invent a natural regularization scheme conserving chirality which falls in between the most general one we describe in § 3 and the very special one of § 7, though we have not yet been able to do this. It will be necessary to do this before rescattering corrections to the π–π scattering lengths can be unambiguously calculated.

It is not unreasonable to expect that our method (omissions of terms corresponding to tadpoles) may be extended to higher orders, in that by it the Ward identities may be verified formally and then corrections arising from regularization calculated. The complexity of the formulae involved is so great however that we have been unable to check this. Even less are we able to say whether or not the corrections can be made to vanish.

Let us now relate our results to superpropagator methods. Such methods have only been developed in any detail for massless theories. The superpropagator depends on an arbitrary analytic function. This corresponds to the arbitrariness in the definition of Δ^n; in other words to the more restrictive regularization scheme used in § 7. Therefore such an approach will violate the Ward identities (though not the Adler condition which can easily be satisfied in the massless case).

We conclude that to obtain a regularized theory consistent with chiral invariance it is necessary to base the superpropagator expressions on perturbation contributions, calculated without performing explicit differentiation on internal lines.

In other field coordinates than those of the σ model, the vector current Green functions can become as complicated as the axial vector ones, even in the single loop case. We have not investigated in detail if it is possible to perform regularization consistent with chiral invariance in these cases; in the massive case this can be enforced by requiring all the formal identities for the products of Δ, Δ_μ, ... to be true, although this may not be necessary.

Because the possibility of satisfying the Ward identities has not been fully established by our work, and we have models in which it is definitely violated, we must be prepared for their violation in nature. This is a similar possibility to that arising in the Adler triangle anomaly in QED (Adler 1969). Our results specify definite ways in which the Ward identities may be violated, and so should allow predictions to be made as to corrections to the Goldberger–Treiman relation (Pagels and Zepeda 1971) and to other results obtained using them.

Appendix. Formal verification of WI involving $\langle A^\mu\phi_1\phi_2\phi_3\phi_4\phi_5\rangle_{+0}$

This WI states that the sum of the diagrams of figure 8 is zero.

The first and second have already been shown to vanish. To see this for the second, notice that in § 4 the form of $\delta\mathscr{L}/\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3$ is not used, and the proof goes through with $\delta\mathscr{L}/\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_2\,\delta\phi_3\,\delta\phi_4\,\delta\phi_5$ instead.

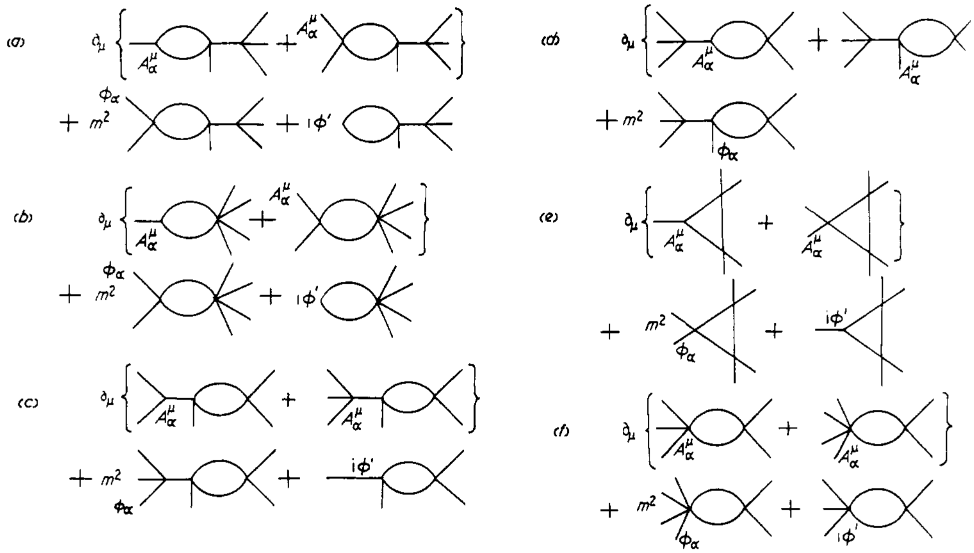

Figure 8. Terms of a Ward identity to order λ^5.

We write the rest as, respectively:

$$-\tfrac{1}{2}\left\{\partial_\mu \frac{\delta A^\mu_\alpha}{\delta\phi_1\,\delta\phi_2\,\delta\phi_\delta}+\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_{\alpha'}\,\delta\phi_\delta}+\mathrm{i}\left(\frac{\delta\phi'}{\delta\phi_\delta\,\delta\phi_2}\,\delta^\alpha_1|_{\delta_1}+1\to 2\right)\right\}$$

$$\times\frac{\delta\mathscr{L}}{\delta\phi_\delta\,\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_3}\,\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}$$

$$+\text{twenty nine permutations of 1, 2, 3, 4 and 5} \tag{52}$$

$$-\tfrac{1}{2}\left(\partial_\mu \frac{\delta A^\mu_\alpha}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_\delta}+\frac{\delta\mathscr{L}}{\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_\delta}\right)$$

$$\times\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_\delta}\,\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}+\text{nine permutations} \tag{53}$$

$$-\tfrac{1}{2}\left(\partial_\mu \frac{\delta A^\mu_\alpha}{\delta\phi_\gamma\,\delta\phi_\delta\,\delta\phi_1}+\frac{\delta\mathscr{L}}{\delta\phi_{\alpha'}\,\delta\phi_\gamma\,\delta\phi_\delta\,\delta\phi_1}+\mathrm{i}\frac{\delta\phi^1}{\delta\phi_\delta\,\delta\phi_\gamma}\right)$$

$$\times\frac{\delta\mathscr{L}}{\delta\phi_\delta\,\delta\phi_\beta\,\delta\phi_2\,\delta\phi_3}\,\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}+\text{twenty nine permutations} \tag{54}$$

$$\frac{\mathrm{i}}{2}\left\{\partial_\mu \frac{\delta A^\mu_\alpha}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_\beta\,\delta\phi_\gamma}+\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_\gamma}+\mathrm{i}\left(\frac{\delta\phi^1}{\delta\phi_1\,\delta\phi_2\,\delta\phi_\beta\,\delta\phi_\gamma}\,\delta^\alpha_3|_{\delta_3}\right.\right.$$

$$\left.\left.+1\to 2\to 3\to 1+1\to 3\to 2\to 1\right)\right\}\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}$$

$$+\text{nine permutations.} \tag{55}$$

The meaning of the prime in $\phi_{\alpha'}$ is given after equation (15), and of $|_{\delta_1}$ etc after (17). Our procedure is as described in § 3. One application not encountered in § 4 arises from the triangle diagrams. When for example the line between x and y in figure 9 is $\Delta^{\mu\mu}(x-y)$ we replace it by $-m^2\,\Delta(x-y)-\mathrm{i}\,\delta(x-y)$. The second term may occur in something like $\delta(x-y)\,\Delta(x-z)\,\Delta^{\nu\nu}(y-z)$. This we replace by

$$\delta(x-y)\,\Delta(x-z)\{-m^2\,\Delta(y-z)\}.$$

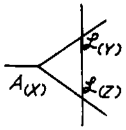

Figure 9. Triangle diagram contributing to the Ward identity for $\langle A^\mu\phi_1\phi_2\phi_3\phi_4\phi_5\rangle_{+0}$.

To calculate (52) we need the following:

$$\frac{\delta A^\mu_\alpha}{\delta\phi_1\,\delta\phi_2\,\delta\phi_\delta} = -\delta^\alpha_{1\mu}\,\delta^\delta_2-\delta^\alpha_{2\mu}\,\delta^\delta_1-\delta^\alpha_{\delta\mu}\,\delta^1_2+\delta^\alpha_1(\delta^{\delta\mu}_{2\mu}+\delta^{\delta\mu}_{2\mu})+\delta^\alpha_2(\delta^{\delta\mu}_1+\delta^\delta_{1\mu})+\delta^\alpha_\delta(\delta^{1\mu}_2+\delta^1_{2\mu}) \tag{56}$$

so

$$\partial_\mu\frac{\delta A^\mu_\alpha}{\delta\phi_1\,\delta\phi_2\,\delta\phi_\delta}$$
$$= 2(\delta^\alpha_1\,\delta^{\delta\mu}_{2\mu}+\delta^\alpha_2\,\delta^{\delta\mu}_{1\mu}+\delta^\alpha_\delta\,\delta^{1\mu}_{2\mu})-m^2(\delta^\alpha_1\,\delta^\delta_2+\delta^\alpha_2\,\delta^\delta_1+\delta^\alpha_\delta\,\delta^1_2)$$
$$+\mathrm{i}|(\delta^\alpha_1\,\delta^\delta_2-\delta^\alpha_2\,\delta^\delta_1-\delta^\alpha_\delta\,\delta^2_1)|_{\delta_1}+1\to 2\to\delta\to 1+1\to\delta\to 2\to 1| \tag{57}$$

$$\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_{\alpha'}\,\delta\phi_\delta}$$
$$= -2(\delta^\alpha_1\,\delta^{\delta\mu}_{2\mu}+\delta^\alpha_2\,\delta^{\delta\mu}_{1\mu}+\delta^\alpha_\delta\,\delta^{1\mu}_{2\mu})+m^2(\delta^\alpha_1\,\delta^\delta_2+\delta^\alpha_2\,\delta^\delta_1+\delta^\alpha_\delta\,\delta^1_2)$$
$$+\mathrm{i}\{(\delta^\alpha_2\,\delta^1_\delta+\delta^\alpha_\delta\,\delta^1_2)|_{\delta_1}+1\to 2\to\delta\to 1+1\to\delta\to 2\to 1\} \tag{58}$$

$$\frac{\delta\phi^1}{\delta\phi_\delta\,\delta\phi_2} = -\delta^\delta_2 \tag{59}$$

so that (52) is equal to

$$-\frac{\mathrm{i}}{2}\delta^1_2\frac{\delta\mathscr{L}}{\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_3}\,\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}+\text{twenty nine permutations.} \tag{60}$$

These also tell us that (53) is

$$-\frac{\mathrm{i}}{2}\delta^\beta_\gamma\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_{\alpha'}}\,\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}+\text{nine permutations} \tag{61}$$

and that (54) is

$$-\mathrm{i}\delta^1_\gamma\frac{\delta\mathscr{L}}{\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_2\,\delta\phi_3}\,\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5}+\text{twenty nine permutations.} \tag{62}$$

To calculate (55) we need

$$\frac{\delta A_\alpha^\mu}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_\beta\,\delta\phi_\gamma}$$

$$= [-\delta^\alpha_{1\mu}(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)-(\delta^\alpha_{\beta\mu}\,\delta^1_\gamma+\delta^\alpha_{\gamma\mu}\,\delta^1_\beta)\,\delta^2_3$$
$$+\delta^\alpha_1\{(\delta^{\beta\mu}_\gamma+\delta^\beta_{\gamma\mu})\,\delta^2_3+\delta^\beta_\gamma(\delta^{2\mu}_3+\delta^2_{3\mu})+(\delta^{\beta\mu}_2+\delta^\beta_{2\mu})\,\delta^\gamma_3+\delta^\beta_3(\delta^{\gamma\mu}_2+\delta^\gamma_{2\mu})$$
$$+(\delta^{\beta\mu}_3+\delta^\beta_{3\mu})\,\delta^\gamma_2+\delta^\beta_2(\delta^{\gamma\mu}_3+\delta^\gamma_{3\mu})\}+\{\delta^\alpha_\beta(\delta^\mu+\delta^1_{\gamma\mu})+\delta^\alpha_\gamma(\delta^{1\mu}_\beta+\delta^1_{\beta\mu})\}\,\delta^2_3$$
$$+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)(\delta^{2\mu}_3+\delta^2_{3\mu})]+1\to2\to3\to1+1\to3\to2\to1 \tag{63}$$

giving

$$\partial_\mu\frac{\delta A_\alpha^\mu}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_\beta\,\delta\phi_\gamma}$$

$$=[2\delta^\alpha_1\{\delta^{\beta\mu}_{\gamma\mu}\,\delta^2_3+\delta^\beta_\gamma\,\delta^{2\mu}_{3\mu}+(\delta^{\beta\mu}_\gamma+\delta^\beta_{\gamma\mu})(\delta^{2\mu}_3+\delta^2_{3\mu})+\delta^{\beta\mu}_{2\mu}\,\delta^\gamma_3+\delta^{\gamma\mu}_{2\mu}\,\delta^\beta_3+\delta^{\beta\mu}_{3\mu}\,\delta^\gamma_2+\delta^{\gamma\mu}_{3\mu}\delta^\beta_2$$
$$+\delta^{\beta\mu}_2\,\delta^{\gamma\mu}_3+\delta^{\beta\mu}_3\,\delta^{\gamma\mu}_2+\delta^\beta_{2\mu}\,\delta^\gamma_{3\mu}+\delta^\beta_{3\mu}\delta^\gamma_{2\mu}+\delta^{\beta\mu}_2\,\delta^\gamma_{3\mu}+\delta^{\gamma\mu}_2\,\delta^\beta_{3\mu}+\delta^{\beta\mu}_3\,\delta^\gamma_{2\mu}+\delta^{\gamma\mu}_3\,\delta^\beta_{2\mu}\}$$
$$+2(\delta^\alpha_\beta\,\delta^{1\mu}_{\gamma\mu}+\delta^\alpha_\gamma\,\delta^{1\mu}_{\beta\mu})\,\delta^2_3+2(\delta^\alpha_\beta\,\delta^1_{\gamma\mu}+\delta^\alpha_\gamma\,\delta^1_{\beta\mu})(\delta^{2\mu}_3+\delta^2_{3\mu})$$
$$+2(\delta^\alpha_\beta\,\delta^{1\mu}_\gamma+\delta^\alpha_\gamma\,\delta^{1\mu}_\beta)(\delta^{2\mu}_3+\delta^2_{3\mu})+2(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^{2\mu}_{3\mu}$$
$$-3m^2\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}$$
$$+\mathrm{i}\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)-(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}|_{\delta_1}$$
$$-\mathrm{i}\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}|_{\delta_2+\delta_3}]$$
$$+1\to2\to3\to1+1\to3\to2\to1 \tag{64}$$

$$\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_\gamma}$$

$$=[-2\delta^\alpha_1\{2\delta^{\beta\mu}_{\gamma\mu}\,\delta^2_3+2\delta^\beta_\gamma\,\delta^{2\mu}_{3\mu}+(\delta^{\beta\mu}_\gamma+\delta^\beta_{\gamma\mu})(\delta^{2\mu}_3+\delta^2_{3\mu})$$
$$+2\delta^{\beta\mu}_{2\mu}+2\delta^{\gamma\mu}_{2\mu}\,\delta^\beta_3+2\delta^{\beta\mu}_{3\mu}\,\delta^\gamma_2+2\delta^{\gamma\mu}_{3\mu}\,\delta^\beta_2+\delta^{\beta\mu}_2\,\delta^{\gamma\mu}_3+\delta^{\beta\mu}_3\,\delta^{\gamma\mu}_2+\delta^\beta_{2\mu}\,\delta^\gamma_{3\mu}+\delta^\beta_{3\mu}\,\delta^\gamma_{2\mu}$$
$$+\delta^{\beta\mu}_2\,\delta^\gamma_{3\mu}+\delta^{\gamma\mu}_2\,\delta^\beta_{3\mu}+\delta^{\beta\mu}_3\,\delta^\gamma_{2\mu}+\delta^{\gamma\mu}_3\,\delta^\beta_{2\mu}\}-4(\delta^\alpha_\beta\,\delta^{1\mu}_{\gamma\mu}+\delta^\alpha_\gamma\,\delta^{1\mu}_{\beta\mu})\,\delta^2_3$$
$$-2(\delta^\alpha_\beta\,\delta^1_{\gamma\mu}+\delta^\alpha_\gamma\,\delta^1_{\beta\mu})(\delta^{2\mu}_3+\delta^2_{3\mu})-2(\delta^\alpha_\beta\,\delta^{1\mu}_\gamma+\delta^\alpha_\gamma\,\delta^{1\mu}_\beta)(\delta^{2\mu}_3+\delta^2_{3\mu})$$
$$-4(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_{3\mu}+5m^2\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)$$
$$+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}+2\mathrm{i}\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}|_{\delta_2+\delta_3}]$$
$$+1\to2\to3\to1+1\to3\to2\to1 \tag{65}$$

$$\frac{\delta\phi^1}{\delta\phi_1\,\delta\phi_2\,\delta\phi_\beta\,\delta\phi_\gamma}=-(\delta^\beta_\gamma\,\delta^1_2+\delta^\beta_1\,\delta^\gamma_2+\delta^\beta_2\,\delta^\gamma_1) \tag{66}$$

so that (55) is

$$\frac{\mathrm{i}}{2}[-2\delta^\alpha_1\{\delta^{\beta\mu}_{\gamma\mu}\,\delta^2_3+\delta^\beta_\gamma\,\delta^{2\mu}_{3\mu}+\delta^{\beta\mu}_{2\mu}\,\delta^\gamma_3+\delta^{\gamma\mu}_{2\mu}\,\delta^\beta_3+\delta^{\beta\mu}_{3\mu}\,\delta^\gamma_2+\delta^{\gamma\mu}_{3\mu}\,\delta^\beta_2\}$$
$$-2(\delta^\alpha_\beta\,\delta^{1\mu}_{\gamma\mu}+\delta^\alpha_\gamma\,\delta^{1\mu}_{\beta\mu})\,\delta^2_3-2(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^{2\mu}_{3\mu}$$
$$+2m^2\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}$$

$$+\mathrm{i}\{(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}|_{\delta_1}+\mathrm{i}\{\delta^\alpha_1(\delta^\beta_\gamma\,\delta^2_3+\delta^\beta_2\,\delta^\gamma_3+\delta^\beta_3\,\delta^\gamma_2)$$
$$+(\delta^\alpha_\beta\,\delta^1_\gamma+\delta^\alpha_\gamma\,\delta^1_\beta)\,\delta^2_3\}|_{\delta_2+\delta_3}]\frac{\delta\mathscr{L}}{\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_3\,\delta\phi_5}+\text{twenty nine permutations.} \tag{67}$$

Finally, if one adds (60), (61) and (62), using the explicit expressions for

$$\frac{\delta\mathscr{L}}{\delta\phi_1\,\delta\phi_2\,\delta\phi_3\,\delta\phi_{\alpha'}},\qquad\frac{\delta\mathscr{L}}{\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_2\,\delta\phi_3},\qquad\frac{\delta\mathscr{L}}{\delta\phi_{\alpha'}\,\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_3},$$

but without mobilizing $\delta\mathscr{L}/\delta\phi_\beta\,\delta\phi_\gamma\,\delta\phi_4\,\delta\phi_5$, one finds minus (67). Therefore formally the sum of the diagrams of figure 8 is in fact zero, as required by the WI.

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